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On *p.p.* structural matrix rings

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ABSTRACT

A ring is called a left *p.p.* ring if every principal left ideal is projective. The objective here is to completely determine the left *p.p.* structural matrix rings over a von Neumann regular ring.

1. Introduction and preliminary

A ring is called a left *p.p.* ring if every principal left ideal is projective. These rings seem to have been first introduced by Hattori [2] in 1960, and are the topic of many publications. This work is motivated by the following two results: Small [5] proved that a ring R is left semihereditary (i.e., every finitely generated left ideal is projective) if and only if every matrix ring over R is a left *p.p.* ring; Nicholson [4] showed that a ring R is (von Neumann) regular if and only if every upper triangular matrix ring over R is a left *p.p.* ring. Thus if R is a regular ring, then all matrix rings and upper triangular matrix rings over R are *p.p.* rings; and the converse holds too. Because matrix rings and upper triangular matrix rings are two special cases of structural matrix rings, we are motivated to ask: Is every structural matrix ring over a regular ring a left *p.p.* ring? If not, then which structural matrix rings over a regular ring are left *p.p.* rings? As seen later, the answer to the first question is "No". In this paper, we will completely

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determine the structural matrix rings over a regular ring which are left $p.p.$ rings and, as a consequence, a new family of left $p.p.$ rings is obtained.

Throughout, R is an associative ring with identity and modules are unitary left modules. For integers m and n, if X is a subset of a ring R, then the set of all $m \times n$ matrices with entries in X is denoted by $\mathbb{M}_{m\times n}(X)$. Thus $\mathbb{M}_{m\times n}(R)$ denotes the set of all $m \times n$ matrices over R. We denote by $\mathbb{M}_n(R)$ the $n \times n$ matrix ring over R, and by $\mathbb{T}_n(R)$ the $n \times n$ upper triangular matrix ring over R. For a left module M over R, the left annihilator of an element x of M in R is denoted by $\mathbf{l}_R(x)$ or simply by $\mathbf{l}(x)$. If R, S are rings and M is an (R, S)-bimodule, then $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ denotes the formal triangular matrix ring. The permutation group on the set $\{1, 2, \ldots, n\}$ is denoted by $\mathrm{Sym}(n)$.

For a binary relation θ on $\{1, 2, \ldots, n\}$, let $B = (b_{ij})$ be the $n \times n$ Boolean matrix where $b_{ij} = 1$ if $(i, j) \in \theta$ and $b_{ij} = 0$ if $(i, j) \notin \theta$. Thus, $\theta \mapsto B$ gives a natural bijection between the set of the binary relations on $\{1, 2, \ldots, n\}$ and the set of all $n \times n$ Boolean matrices. For a binary relation θ on $\{1, 2, \ldots, n\}$ with the corresponding Boolean matrix $B = (b_{ij})$, there is an associated additively closed subset $\mathbb{M}_n(B, R)$ $\big($also denoted by $\mathbb{M}_n(\theta, R)\big)$ of $\mathbb{M}_n(R)$, where

$$\begin{aligned} \mathbb{M}_n(B, R) &= \{(a_{ij}) \in \mathbb{M}_n(R) : b_{ij} = 0 \Rightarrow a_{ij} = 0\} \\ &= \{(a_{ij}) \in \mathbb{M}_n(R) : (i, j) \notin \theta \Rightarrow a_{ij} = 0\}. \end{aligned}$$

A quasi-order Boolean matrix is a Boolean matrix $B = (b_{ij})$ which is reflexive (i.e., $b_{ii} = 1$ for all i) and transitive (i.e., $b_{ij} = 1 = b_{jk}$ implies $b_{ik} = 1$ for all i, j, k). It is clear that B is transitive if and only if $\mathbb{M}_n(B, R)$ is multiplicatively closed in $\mathbb{M}_n(R)$, and that B is reflexive if and only if $\mathbb{M}_n(B, R)$ contains the identity matrix I_n.

Definition 1.1. For an $n \times n$ quasi-order Boolean matrix B, the subring $\mathbb{M}_n(B, R)$ of $\mathbb{M}_n(R)$ is called the structural matrix ring over R associated with B.

An $n \times n$ quasi-order Boolean matrix B is called blocked triangular (see [1]) if it is of the form $\begin{pmatrix} B_{11} & B_{12} & \ldots & B_{1k} \\ 0 & B_{22} & \ldots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B_{kk} \end{pmatrix}$, where for every $i \le j$, B_{ij} is an $n_i \times n_j$ (Boolean) matrix with all its entries equal, and $n_1 + \cdots + n_k = n$. Since we assume B to be quasi-order, every entry of B_{ii} is 1 for $i = 1, \ldots, k$. If every entry of B_{ij} is 1 for all $i \le j$, then B is called complete blocked triangular [1].

Example 1.2. Let $B = (b_{ij})$ be an $n \times n$ Boolean matrix.

(1) If $b_{ij} = 1$ for all $i \le j$ and $b_{ij} = 0$ for all $i > j$, then $\mathbb{M}_n(B, R) = \mathbb{T}_n(R)$.
(2) If $b_{ii} = 1$ for all i and $b_{ij} = 0$ for all $i \neq j$, then $\mathbb{M}_n(B, R)$ is isomorphic to the direct product of n copies of R.
(3) If $b_{ij} = 1$ for all i and j, then $\mathbb{M}_n(B, R) = \mathbb{M}_n(R)$.
(4) A complete blocked triangular matrix ring [3] is a ring of the form

$$\begin{pmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1\times n_2}(R) & \ldots & \mathbb{M}_{n_1\times n_k}(R) \\ 0 & \mathbb{M}_{n_2}(R) & \ldots & \mathbb{M}_{n_2\times n_k}(R) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbb{M}_{n_k}(R) \end{pmatrix}, \quad k \geqslant 1.$$

Clearly, a complete blocked triangular matrix ring over R is a structural matrix ring over R associated with some complete blocked triangular Boolean matrix. A structural matrix ring $\mathbb{M}_n(B, R)$ is called blocked triangular if B is blocked triangular.

(5) The subring $T := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(R) : a + c = b + d \right\}$ of $\mathbb{M}_2(R)$ is not a structural matrix ring, but it is isomorphic to a 2×2 structural matrix ring over R. In fact, $T \overset{\varphi}{\cong} \mathbb{T}_2(R)$, where $\varphi(X) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} X \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ for all $X \in T$.

If $A = (a_{ij})$ is an $n \times n$ matrix and $\sigma \in \mathrm{Sym}(n)$, let A^σ denote the $n \times n$ matrix whose $(\sigma(i), \sigma(j))$-entry is a_{ij} for all $i, j \in \{1, 2, \ldots, n\}$. Given an $n \times n$ quasi-order Boolean matrix B and $\sigma \in \mathrm{Sym}(n)$, B^σ is also a quasi-order Boolean matrix, and $\mathbb{M}_n(B, R) \cong \mathbb{M}_n(B^\sigma, R)$ via the isomorphism $A \mapsto A^\sigma$. Moreover, there exists a permutation $\sigma \in \mathrm{Sym}(n)$ such that B^σ is blocked triangular. This is the idea used in [1] to prove the next theorem, which first appeared in [6] and which will be frequently used in the rest of this paper.

Theorem 1.3 [6]. *Every structural matrix ring over R is isomorphic to a blocked triangular matrix ring over R. Precisely, if B is an $n \times n$ quasi-order Boolean matrix, then B^σ is blocked triangular for some $\sigma \in \mathrm{Sym}(n)$ and so*

$$\mathbb{M}_n(B, R) \cong \mathbb{M}_n(B^\sigma, R) = \begin{pmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1 \times n_2}(X_{12}) & \ldots & \mathbb{M}_{n_1 \times n_k}(X_{1k}) \\ 0 & \mathbb{M}_{n_2}(R) & \ldots & \mathbb{M}_{n_2 \times n_k}(X_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbb{M}_{n_k}(R) \end{pmatrix},$$

where $X_{ij} = 0$ or $X_{ij} = R$ for all i, j with $1 \le i < j \le k$.

For convenience, the number k in Theorem 1.3 is called the block size of the blocked triangular matrix B^σ and also of the blocked triangular matrix ring $\mathbb{M}_n(B^\sigma, R)$. Given a quasi-order relation θ on the set $\{1, 2, \ldots, n\}$, it naturally gives rise to an equivalence relation $\sim$ on $\{1, 2, \ldots, n\}$ defined by $i \sim j$ if and only if both $(i, j) \in \theta$ and $(j, i) \in \theta$. If B is the Boolean matrix determined by θ and $\sigma \in \mathrm{Sym}(n)$ is the permutation such that B^σ is a blocked triangular Boolean matrix, then the numbers $\{n_i : i = 1, 2, \ldots, k\}$ in Theorem 1.3 are actually the cardinalities of the equivalence classes induced by $\sim$.

We refer the readers to the articles [1,6–9] for further background material on structural matrix rings.

2. Main results

Given an $n \times n$ quasi-order Boolean matrix B, let B^σ be a blocked triangular Boolean matrix where $\sigma \in \mathrm{Sym}(n)$. If R is a regular ring, then the left $p.p.$ property of $\mathbb{M}_n(B, R)$ is uniquely determined by the graph of B^σ. Our main theorem can be stated as follows.

Theorem 2.1. *Let R be a regular ring and B be an $n \times n$ quasi-order Boolean matrix with a blocked triangular Boolean matrix B^σ, where $\sigma \in \mathrm{Sym}(n)$. Write*

$$\mathbb{M}_n(B^\sigma, R) = \begin{pmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1 \times n_2}(X_{12}) & \ldots & \mathbb{M}_{n_1 \times n_k}(X_{1k}) \\ 0 & \mathbb{M}_{n_2}(R) & \ldots & \mathbb{M}_{n_2 \times n_k}(X_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbb{M}_{n_k}(R) \end{pmatrix},$$

where $X_{ij} = 0$ or $X_{ij} = R$ for all i, j with $1 \le i < j \le k$.

(1) *If* $k \le 3$*, then* $\mathbb{M}_n(B, R)$ *is a left p.p. ring.*
(2) *If* $k \ge 4$*, then* $\mathbb{M}_n(B, R)$ *is not left p.p. if and only if there exist* $1 \leqslant l_1 < l_2 < l_3 < l_4 \leqslant k$ *such that* $X_{l_2l_3} = 0, X_{l_1l_2} = X_{l_1l_3} = X_{l_2l_4} = X_{l_3l_4} = R$*.*

We remark that, for $k = 4$, $X_{12} = X_{24} = R$ implies $X_{14} = R$; thus $\mathbb{M}_n(B, R)$ is not left *p.p.* $\Longleftrightarrow$ "$X_{23} = 0$ and $X_{12} = X_{13} = X_{24} = X_{34} = R$" $\Longleftrightarrow$ "$X_{23} = 0$ and $X_{12} = X_{13} = X_{14} = X_{24} = X_{34} = R$".

In view of Theorem 1.3, Theorem 2.1 can be restated as the following: A structural matrix ring T over a regular ring R is not a left *p.p.* ring if and only if T is isomorphic to a blocked triangular matrix ring

$$\begin{pmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1\times n_2}(X_{12}) & \dots & \mathbb{M}_{n_1\times n_k}(X_{1k}) \\ 0 & \mathbb{M}_{n_2}(R) & \dots & \mathbb{M}_{n_2\times n_k}(X_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{M}_{n_k}(R) \end{pmatrix},$$

where $X_{ij} = 0$ or $X_{ij} = R$ for all i, j with $1 \le i < j \le k$, such that $X_{l_2l_3} = 0$ and $X_{l_1l_2} = X_{l_1l_3} = X_{l_2l_4} = X_{l_3l_4} = R$ for some $1 \leqslant l_1 < l_2 < l_3 < l_4 \leqslant k$.

In [4, Theorem 4], Nicholson proved that, for $n \ge 2$, $\mathbb{T}_n(R)$ is a left *p.p.* ring if and only if R is a regular ring. This is an immediate consequence of the next corollary. A complete set of $n \times n$ matrix units is denoted by $\{E_{ij} : 1 \le i, j \le n\}$.

Corollary 2.2. *Let R be a ring, and T be a complete blocked triangular matrix ring over R with block size greater than or equal to 2. Then R is a regular ring if and only if T is a left p.p. ring.*

Proof. ($\Longrightarrow$). This is by Theorem 2.1.

($\Longleftarrow$). Let T be an $n \times n$ complete blocked triangular matrix ring over R. Since the block size of T is greater than or equal to 2, we have $n \ge 2$. For $a \in R$, let $A = aE_{11} - E_{1n}$ and $B = E_{1n} + aE_{nn}$. Then $A, B \in T$ and $AB = 0$. Since T is a left *p.p.* ring, there exists an idempotent E of T such that $A \in \mathbf{l}_T(B) = TE$. Thus, $A = AE$ and $EB = 0$. Write $E = (e_{ij})$. Then it follows that $-1 = ae_{1n} - e_{nn}$ and $e_{nn}a = 0$. So $a = 1 \cdot a = (-ae_{1n} + e_{nn})a = a(-e_{1n})a$. This shows that R is a regular ring. $\square$

3. Proof of Theorem 2.1

A left module is called a *p.p.* module if every principal submodule is projective. It is well-known that a left R-module M is *p.p.* if and only if the left annihilator $\mathbf{l}_R(x)$ is generated by an idempotent for all $x \in M$. In particular, R is a left *p.p.* ring if and only if $\mathbf{l}(a)$ is generated by an idempotent for all $a \in R$. We need the following lemmas for the proof of Theorem 2.1.

Lemma 3.1 [4, Corollary 1]**.** *Let S be a regular ring. Then* $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *is a left p.p. ring if and only if R is a left p.p. ring and* ${}_RM$ *is a p.p. module.*

Lemma 3.2 [10]**.** *If* $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *is a left p.p. ring, then R and S are also left p.p. rings.*

Lemma 3.3 [4, Theorem 1]**.** *A ring R is a left p.p. ring if and only if every finitely generated free left R-module is a p.p. module.*

Lemma 3.4. *(1) Submodules of a p.p. module are p.p. modules.*
(2) Direct sums of p.p. left modules over a left p.p. ring are p.p. modules.

Proof. (1) This is obvious.

(2) Let R be a left $p.p.$ ring and $M = \oplus_{i \in I} M_i$ where each M_i is a $p.p.$ left R-module. To show that M is a $p.p.$ module, we can assume that I is a finite set and further assume that the cardinal of I is 2. So $M = M_1 \oplus M_2$. Let $x \in M$ and write $x = x_1 + x_2$ where $x_1 \in M_1$ and $x_2 \in M_2$. By hypothesis, $\mathbf{l}(x_i) = Re_i$ where $e_i^2 = e_i \in R$ for $i = 1, 2$. Since R is a left $p.p.$ ring, $Re_1 \cap Re_2 = Re$ for some $e^2 = e \in R$ by [4, Lemma 1]. Thus $\mathbf{l}(x) = \mathbf{l}(x_1) \cap \mathbf{l}(x_2) = Re$. This proves that M is a $p.p.$ module. □

For $n \geq 1$, let $R^n = \mathbb{M}_{1\times n}(R)$ and $R_n = \mathbb{M}_{n\times 1}(R)$. Then R_n is a left module over $\mathbb{M}_n(R)$, and hence it is a left module over $\mathbb{M}_n(\theta, R)$. Define $1_n = \begin{pmatrix}1 & 1 & \dots & 1\end{pmatrix}$, $0_n = \begin{pmatrix}0 & 0 & \dots & 0\end{pmatrix} \in R^n$. The argument outlined in the next lemma will be repeatedly used in proving our main result.

Lemma 3.5. (*The* (l, m)*-argument*) *Let B be an $n \times n$ blocked triangular Boolean matrix, and*

$$\mathbb{M}_n(B, R) = \begin{pmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1\times n_2}(X_{12}) & \dots & \mathbb{M}_{n_1\times n_t}(X_{1t}) \\ 0 & \mathbb{M}_{n_2}(R) & \dots & \mathbb{M}_{n_2\times n_t}(X_{2t}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{M}_{n_t}(R) \end{pmatrix},$$

where $t \geq 3$, and $X_{ij} = 0$ or $X_{ij} = R$ for all i, j with $1 \leq i < j \leq t$. If there exist $1 < l < m \leqslant t$ such that $X_{lm} = 0$, $X_{1l} = X_{1m} = R$ and $X_{1i}X_{il} = 0$ for all $1 < i < l$, then the left annihilator of

$$v(l, m) := \begin{pmatrix} 1_{n_1} & 0_{n_2+\dots+n_{l-1}} & 1_{n_l} & 0_{n_{l+1}+\dots+n_{m-1}} & 1_{n_m} & 0_{n_{m+1}+\dots+n_t} \end{pmatrix}^T \in R_n$$

in $\mathbb{M}_n(B, R)$ is not generated by an idempotent. We refer to this argument as the (l, m)-argument.

Proof. Let $S = \mathbb{M}_n(B, R)$ and $v = v(l, m)$. Suppose on the contrary that there exists some $E^2 = E \in S$ such that $SE = \mathbf{l}_S(v)$. For a matrix X, if all the row sums of X are equal, we denote this element of R by $\varsigma(X)$. Then

$$\mathbf{l}_S(v) = \left\{ \begin{pmatrix} A & \cdots & Y_{1l} & \cdots & Y_{1m} & \cdots \\ & \ddots & \vdots & & \vdots & \\ & & B & \cdots & 0 & \cdots \\ & & & \ddots & \vdots & \\ & 0 & & & C & \cdots \\ & & & & & \ddots \end{pmatrix} \in S : \varsigma([A\, Y_{1l}\, Y_{1m}]) = \cdots = \varsigma(B) = \cdots = \varsigma(C) = 0 \right\}.$$

Write $E = \begin{pmatrix} E_1 & \cdots & Y'_{1l} & \cdots & Y'_{1m} & \cdots \\ & \ddots & \vdots & & \vdots & \\ & & E_l & \cdots & 0 & \cdots \\ & & & \ddots & \vdots & \\ & 0 & & & E_m & \cdots \\ & & & & & \ddots \end{pmatrix}$, $E_i^2 = E_i$, $1 \leqslant i \leqslant t$, $\varsigma(E_l) = 0$. Then

$$SE=\left\{\begin{pmatrix} AE_1 & \cdots & AY'_{1l}+Y_{1l}E_l & \cdots & AY'_{1m}+\cdots+Y_{1m}E_m & \cdots \\ & \ddots & \vdots & & \vdots & \\ & & BE_l & \cdots & 0 & \cdots \\ & & & \ddots & \vdots & \\ 0 & & & & CE_m & \cdots \\ & & & & & \ddots \end{pmatrix} : \begin{pmatrix} A & \cdots & Y_{1l} & \cdots & Y_{1m} & \cdots \\ & \ddots & \vdots & & \vdots & \\ & & B & \cdots & 0 & \cdots \\ & & & \ddots & \vdots & \\ 0 & & & & C & \cdots \\ & & & & & \ddots \end{pmatrix} \in S\right\}.$$

Denote the sums of the rows of the matrix $I_{n_1}-E_1$ by $a_1, a_2, \ldots, a_{n_1}$, respectively. Let

$$Q_{1l}=\begin{pmatrix} -a_1 & 0 & \cdots & 0 \\ -a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ -a_{n_1} & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{M}_{n_1\times n_l}(R).$$

It is clear that

$$\begin{pmatrix} I_{n_1}-E_1 & \cdots & Q_{1l} & \cdots & 0 & \cdots \\ & \ddots & \vdots & & \vdots & \\ & & 0 & \cdots & 0 & \cdots \\ & & & \ddots & \vdots & \\ & & & & 0 & \cdots \\ & & & & & \ddots \end{pmatrix}$$

belongs to $\mathbf{l}_S(v)=SE$, so $I_{n_1}-E_1=AE_1$ for some $A\in\mathbb{M}_{n_1}(R)$. Thus, $I_{n_1}-E_1=(I_{n_1}-E_1)^2=(AE_1)(I_{n_1}-E_1)=0$, which gives that $E_1=I_{n_1}$. It also follows from $\mathbf{l}_S(v)=SE$ that $\varsigma(BE_l)=0$ for all $B\in\mathbb{M}_{n_l}(R)$, and this in turn implies that $\varsigma(PE_l)=0$ for all $P\in\mathbb{M}_{n_1\times n_l}(R)$.

Fix some $\begin{pmatrix} A & \cdots & AY'_{1l}+Y_{1l}E_l & \cdots & AY'_{1m}+\cdots+Y_{1m}E_m & \cdots \\ & \ddots & \vdots & & \vdots & \\ & & BE_l & \cdots & 0 & \cdots \\ & & & \ddots & \vdots & \\ 0 & & & & CE_m & \cdots \\ & & & & & \ddots \end{pmatrix} \in SE$, and let

$$W=\begin{pmatrix} A & \cdots & AY'_{1l}+Y_{1l}E_l+G_{11} & \cdots & AY'_{1m}+\cdots+Y_{1m}E_m-H_{11} & \cdots \\ & \ddots & \vdots & & \vdots & \\ & & BE_l & \cdots & 0 & \cdots \\ & & & \ddots & \vdots & \\ 0 & & & & CE_m & \cdots \\ & & & & & \ddots \end{pmatrix},$$

where $G_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{M}_{n_1 \times n_l}(R)$ and $H_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{M}_{n_1 \times n_m}(R)$. Clearly $W \in \mathbf{l}_S(v)$. Now we prove $W \notin SE$. Assume that $W \in SE$. Then there exist $A' \in \mathbb{M}_{n_1}(R)$ and $Z_{1l} \in \mathbb{M}_{n_1 \times n_l}(R)$ such that $A = A'E_1$ and $AY'_{1l} + Y_{1l}E_l + G_{11} = A'Y'_{1l} + Z_{1l}E_l$. Since $E_1 = I_{n_1}$, one obtains that $A = A'$ and hence $Y_{1l}E_l + G_{11} = Z_{1l}E_l$. Denote the sum of the first row of a matrix X by $\varsigma_1(X)$. Then $\varsigma_1(Y_{1l}E_l + G_{11}) = \varsigma_1(Y_{1l}E_l) + \varsigma_1(G_{11}) = \varsigma_1(G_{11}) = 1 \neq 0 = \varsigma_1(Z_{1l}E_l)$ (as $\varsigma(PE_l) = 0$ for all $P \in \mathbb{M}_{n_1 \times n_l}(R)$). This contradiction shows that $W \notin SE$ but $W \in \mathbf{l}_S(v)$. Hence $\mathbf{l}_S(v)$ is not generated by an idempotent. $\square$

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Theorem 1.3, we can assume that $B = B^\sigma$, and let $T = \mathbb{M}_n(B, R)$.

(1) If $k = 1$, then $T = \mathbb{M}_n(R)$ is a regular ring and hence a left $p.p.$ ring.

If $k = 2$, then $T = \begin{pmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1 \times n_2}(X_{12}) \\ 0 & \mathbb{M}_{n_2}(R) \end{pmatrix}$. It can be seen that $\mathbb{M}_{n_1 \times n_2}(X_{12})$ is a finitely generated projective left $\mathbb{M}_{n_1}(R)$-module, and thus a $p.p.$ module by Lemmas 3.3 and 3.4. Since $\mathbb{M}_{n_1}(R)$ and $\mathbb{M}_{n_2}(R)$ are regular, T is a left $p.p.$ ring by Lemma 3.1.

If $k = 3$, write $T = \begin{pmatrix} S & M \\ 0 & \mathbb{M}_{n_3}(R) \end{pmatrix}$, where $S = \begin{pmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1 \times n_2}(X_{12}) \\ 0 & \mathbb{M}_{n_2}(R) \end{pmatrix}$ and $M = \begin{pmatrix} \mathbb{M}_{n_1 \times n_3}(X_{13}) \\ \mathbb{M}_{n_2 \times n_3}(X_{23}) \end{pmatrix}$. Since S is already a left $p.p.$ ring and $\mathbb{M}_{n_3}(R)$ is a regular ring, we need only to show that M is a $p.p.$ left S-module by Lemma 3.1. Let $M_1 = \begin{pmatrix} \mathbb{M}_{n_1 \times 1}(X_{13}) \\ \mathbb{M}_{n_2 \times 1}(X_{23}) \end{pmatrix}$. As left S-modules, M is isomorphic to a direct sum of n_3 copies of M_1. So to show that ${}_SM$ is a $p.p.$ module, it suffices to show that M_1 is a $p.p.$ left S-module by Lemma 3.4 (as S is a left $p.p.$ ring). If $X_{12} = 0$, then M_1 is obviously isomorphic to a submodule of the $p.p.$ module ${}_S(S \oplus S)$ and so it is a $p.p.$ module. If $X_{12} = R$, then M_1 is a submodule of the $(n_1 + 1)^{th}$ column of S and hence is isomorphic to a submodule of the $p.p.$ module ${}_SS$. This shows that M_1 is a $p.p.$ left S-module. So (1) holds.

(2) ($\Longleftarrow$). We prove the implication by induction on k. First assume $k = 4$. By hypothesis and the remark made after Theorem 2.1, $X_{23} = 0$ and $X_{12} = X_{13} = X_{14} = X_{24} = X_{34} = R$, so

$$T = \begin{pmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1 \times n_2}(R) & \mathbb{M}_{n_1 \times n_3}(R) & \mathbb{M}_{n_1 \times n_4}(R) \\ 0 & \mathbb{M}_{n_2}(R) & 0 & \mathbb{M}_{n_2 \times n_4}(R) \\ 0 & 0 & \mathbb{M}_{n_3}(R) & \mathbb{M}_{n_3 \times n_4}(R) \\ 0 & 0 & 0 & \mathbb{M}_{n_4}(R) \end{pmatrix}.$$

Write T as a formal triangular matrix ring $T = \begin{pmatrix} S & M \\ 0 & \mathbb{M}_{n_4}(R) \end{pmatrix}$. Since $\mathbb{M}_{n_4}(R)$ is regular, to show that T is not left $p.p.$, we need only to show that ${}_SM$ is not a $p.p.$ module by Lemma 3.1. By Lemma 3.4, it suffices to show that $M_1 := \begin{pmatrix} \mathbb{M}_{n_1 \times 1}(R) \\ \mathbb{M}_{n_2 \times 1}(R) \\ \mathbb{M}_{n_3 \times 1}(R) \end{pmatrix}$ is not a $p.p.$ left S-module because ${}_SM$ is isomorphic to a direct sum of n_4 copies of M_1. Note that $X_{23} = 0, X_{12} = X_{13} = R$, and $v(2, 3) = \left(1_{n_1} 1_{n_2} 1_{n_3}\right)^T \in M_1$. The (2, 3)-argument shows that the left annihilator of $v(2, 3)$ in S is not generated by an idempotent. Hence M_1 is not a $p.p.$ left S-module. So we have proved that T is not left $p.p.$ if $k = 4$.

Now assume that $t \geq 4$ and that T is not left $p.p.$ for all k with $4 \leqslant k \leqslant t$. We next show that T is not left $p.p.$ for $k = t+1$. If $l_4 < k$, then the upper left $(n_1 + \cdots + n_t) \times (n_1 + \cdots + n_t)$ block of T is not a left $p.p.$ ring by induction hypothesis, and hence T is not left $p.p.$ by Lemma 3.2. So we can assume $l_4 = k$. We write T as a formal triangular matrix ring $T := \begin{pmatrix} S & M \\ 0 & \mathbb{M}_{n_{t+1}}(R) \end{pmatrix}$, and proceed with two situations. Note that we already have $l_4 = k = t+1$.

Case 1: $l_2 = t-1$ and $l_3 = t$. That is, $X_{t-1,t} = 0, X_{t-1,t+1} = X_{t,t+1} = R$. If $l_1 > 1$, then the lower right $(n_2 + \cdots + n_{t+1}) \times (n_2 + \cdots + n_{t+1})$ block of T is not a left $p.p.$ ring by induction hypothesis, and hence T is not left $p.p.$ by Lemma 3.2. Thus we can assume $l_1 = 1$, i.e., $X_{1,t-1} = X_{1t} = R$. We can further assume $X_{i,t-1}X_{it} = 0$ for all i with $2 \leqslant i \leqslant t-2$. (In fact, if $X_{i,t-1}X_{it} = R$ for some i with $2 \leqslant i \leqslant t-2$, then $X_{i,t-1} = X_{it} = R$. Now $1 < i < t-1 < t < t+1$. Take $l_1 = i, l_2 = t-1, l_3 = t$ and $l_4 = t+1$. Then $X_{l_2 l_3} = 0$ and $X_{l_1 l_2} = X_{l_1 l_3} = X_{l_2 l_4} = X_{l_3 l_4} = R$. Since $l_1 > 1$, the lower right $(n_2 + \cdots + n_{t+1}) \times (n_2 + \cdots + n_{t+1})$ block of T is not a left $p.p.$ ring by induction hypothesis, and hence T is not left $p.p.$ by Lemma 3.2) Moreover, it follows from $X_{1t} = X_{t,t+1} = R$ that $X_{1,t+1} = R$.

Denote $M_1 = \begin{pmatrix} \mathbb{M}_{n_1 \times 1}(X_{1,t+1}) \\ \mathbb{M}_{n_2 \times 1}(X_{2,t+1}) \\ \vdots \\ \mathbb{M}_{n_t \times 1}(X_{t,t+1}) \end{pmatrix}$. Then as left S-modules, M_1 is isomorphic to a direct summand of M. Since $\mathbb{M}_{n_{t+1}}(R)$ is regular, to show that T is not left $p.p.$, we need only to show that M_1 is not a $p.p.$ left S-module by Lemmas 3.1 and 3.4.

For $2 \leq i \leq t-2$, we denote the condition $X_{1i} = X_{i,t-1} = R$ by $P(i)$. We prove the following

Claim. $P(i)$ implies that M_1 is not a $p.p.$ left S-module for all $2 \leq i \leq t-2$.

Proof of Claim. First suppose that $P(2)$ holds, i.e., $X_{12} = X_{2,t-1} = R$. Then $X_{2t} = 0$ (as $X_{2,t-1}X_{2t} = 0$) and $X_{2,t+1} = R$ (as $R = X_{2,t-1}X_{t-1,t+1} \subseteq X_{2,t+1}$). Note that $X_{2t} = 0, X_{12} = X_{1t} = R$, and $v(2,t) = \left(1_{n_1} \ 1_{n_2} \ 0_{n_3+\cdots+n_{t-1}} \ 1_{n_t}\right)^T \in M_1$. So the $(2,t)$-argument shows that the left annihilator of $v(2,t)$ in S is not generated by an idempotent. Hence M_1 is not a $p.p.$ left S-module.

Assume that $2 < q \leq t-2$ and that $P(i)$ implies that M_1 is not a $p.p.$ left S-module for all $2 \leq i < q$. We next show that $P(q)$ implies that M_1 is not a $p.p.$ left S-module.

Since $P(q)$ holds, $X_{1q} = X_{q,t-1} = R$. Then $X_{qt} = 0$ (as $X_{q,t-1}X_{qt} = 0$) and $X_{q,t+1} = R$ (as $R = X_{q,t-1}X_{t-1,t+1} \subseteq X_{q,t+1}$).

If there exists some p with $1 < p < q$ such that $X_{1p}X_{pq} = R$, then $X_{1p} = X_{pq} = R$. So $X_{p,t-1} = R$ (as $X_{pq}X_{q,t-1} \subseteq X_{p,t-1}$). So $P(p)$ holds and hence M_1 is not a $p.p.$ left S-module by induction hypothesis. Thus, we can assume that $X_{1i}X_{iq} = 0$ for all $1 < i < q$. But we have that $X_{qt} = 0$ and $X_{1q} = X_{1t} = R$, and that $v(q,t) = \left(1_{n_1} \ 0_{n_2+\cdots+n_{q-1}} \ 1_{n_q} \ 0_{n_{q+1}+\cdots+n_{t-1}} \ 1_{n_t}\right)^T \in M_1$ (as $X_{1,t+1} = X_{q,t+1} = X_{t,t+1} = R$). So the (q,t)-argument shows that the left annihilator of $v(q,t)$ is S is not generated by an idempotent, and hence M_1 is not a $p.p.$ left S-module. Thus, the claim has been proved by the Induction Principle.

Now come back to the proof. By the claim, to show that M_1 is not a $p.p.$ left S-module we can assume $X_{1i}X_{i,t-1} = 0$ for all i with $2 \leqslant i \leqslant t-2$. Note that $X_{t-1,t} = 0$ and $X_{1,t-1} = X_{1t} = R$, and that $v(t-1,t) = \left(1_{n_1} \ 0_{n_2+\cdots+n_{t-2}} \ 1_{n_{t-1}} \ 1_{n_t}\right)^T \in M_1$ (as $X_{1,t+1} = X_{t-1,t+1} = X_{t,t+1} = R$). So the ($t$-1, t)-argument shows that M_1 is not a $p.p.$ left S-module.

Case 2: $l_2 \leq t-2$. That is, $X_{l_2 l_3} = 0, X_{l_1 l_2} = X_{l_1 l_3} = X_{l_2,t+1} = X_{l_3,t+1} = R$ where $2 \leq l_2 < l_3 \leq t$ and $l_2 \leq t-2$. If $l_1 > 1$, then the lower right $(n_2 + \cdots + n_{t+1}) \times (n_2 + \cdots + n_{t+1})$ block of T is not a left $p.p.$ ring by induction hypothesis, and hence T is not left $p.p.$ by Lemma 3.2. Thus we can assume $l_1 = 1$, i.e., $X_{1l_2} = X_{1l_3} = R$, and can assume $X_{il_2}X_{il_3} = 0$ for all i with $2 \leqslant i \leqslant l_2 - 1$. By arguing as in Case 1 with $t-1$ being replaced by l_2 and t by l_3, we can further assume $X_{1i}X_{il_2} = 0$ for all $2 \leqslant i < l_2$. Noting that $X_{l_2,t+1} = X_{l_3,t+1} = R$ and $X_{1,t+1} = R$ (as $X_{1l_3}X_{l_3,t+1} \subseteq X_{1,t+1}$), we see that $v(l_2, l_3) = \begin{pmatrix} 1_{n_1} & 0_{n_2+\cdots+n_{l_2-1}} & 1_{n_{l_2}} & 0_{n_{l_2+1}+\cdots+n_{l_3-1}} & 1_{n_{l_3}} & 0_{n_{l_3+1}+\cdots+n_t} \end{pmatrix}^T \in M_1$. So the (l_2, l_3)-argument shows that M_1 is not a $p.p.$ left S-module.

(2) ($\Longrightarrow$). Suppose T is not a left $p.p.$ ring. In view of (1), there exists an integer t with $3 \leqslant t < k$ such that the upper left $(n_1 + \cdots + n_t) \times (n_1 + \cdots + n_t)$ block of T is a left $p.p.$ ring, but the upper left $(n_1 + \cdots + n_{t+1}) \times (n_1 + \cdots + n_{t+1})$ block of T is not a left $p.p.$ ring. Denote the second block as $\mathbb{M}_m(B', R)$, where $m = n_1 + \cdots + n_{t+1}$ and B' is the upper left $m \times m$ block of B. Then we need only to show that the necessity of (2) for $\mathbb{M}_m(B', R)$. Therefore, without loss of generality, we can assume that $t = k - 1$. Write $T = \begin{pmatrix} S & M \\ 0 & \mathbb{M}_{n_k}(R) \end{pmatrix}$ and define M_1 as before. Now S is a left $p.p.$ ring. Hence ${}_SM$ is not a $p.p.$ module by Lemma 3.1. Since ${}_SM$ is isomorphic to the direct sum of n_k copies of ${}_SM_1$, ${}_SM_1$ is not a $p.p.$ module by Lemma 3.4. Hence ${}_SM_1 \neq 0$ and it is not a direct sum of columns of S.

First assume all $n_i = 1, 1 \leqslant i \leqslant k-1$, and let C_i be the i^{th} column of S. We now express M_1 as a sum of some columns of S. Since $M_1 \neq 0$, $\Lambda_1 := \{i : X_{ik} = R,\ 1 \leqslant i < k\}$ is not empty and so $q_1 = max(\Lambda_1)$ is well-defined. From the multiplication of matrices in T, we see that, for any $1 \leq i < q_1, X_{i,q_1} = R$ implies $X_{ik} = R$; we hence deduce that $C_{q_1} \subseteq M_1$. But $C_{q_1} \neq M_1$ since M_1 is not a direct sum of columns of S, so $\Lambda_2 := \{i : X_{ik} = R \text{ and } X_{i,q_1} = 0,\ 1 \leqslant i < q_1\}$ is not empty. Let $q_2 = max(\Lambda_2)$. Then $C_{q_2} \subseteq M_1$. If $M_1 = C_{q_1} + C_{q_2}$, we are done; if not, then $\Lambda_3 = \{i : X_{ik} = R \text{ and } X_{i,q_1} = X_{i,q_2} = 0,\ 1 \leqslant i < q_2\}$ is not empty. Let $q_3 = max(\Lambda_3)$ and, again, $C_{q_3} \subseteq M_1$. A simple induction shows that there exists an integer p with $2 \leq p < k$ such that $M_1 = C_{q_1} + C_{q_2} + \cdots + C_{q_p}$ and $q_1 > q_2 > \cdots > q_p$.

The sum $M_1 = C_{q_1} + C_{q_2} + \cdots + C_{q_p}$ is not direct, so

$$0 = x_1 + x_2 + \cdots + x_p,$$

where $x_i \in C_{q_i}$ for $i = 1, \ldots, p$ and the $x_i{}'s$ are not all zero. Without loss of generality, we can assume that $x_i \neq 0$ for all $i = 1, \ldots, p$. By the choice of q_p, we have $X_{q_p,q_j} = 0$ for all $1 \leq j < p$. In particular, the q_p^{th} entry of each x_j (regarding x_j as a column vector), $1 \leqslant j < p$, is 0. Thus, the q_p^{th} entry of x_p is zero too. But since $x_p \neq 0$, there exists an integer l with $1 \leq l < q_p$ such that the l^{th} entry of x_p is not zero. Then by the above equation, there exists some $1 \leqslant r < p$ such that the l^{th} entry of x_r is nonzero. These show that $X_{l,q_p} = X_{l,q_r} = R$. By the choice of q_p and $q_r, X_{q_p,k} = X_{q_r,k} = R$. Moreover, $X_{q_p,q_r} = 0$. Since $1 \leqslant l < q_p < q_r < k$, we have proved the necessity of (2) by taking $l_1 = l$, $l_2 = q_p$, $l_3 = q_r$ and $l_4 = k$.

As to arbitrary n_i $(1 \leqslant i \leqslant k - 1)$, if we let $C_1, C_2, \ldots, C_{k-1}$ be the first, the $(n_1 + 1)^{th}, \ldots,$ the $(n_{k-2} + 1)^{th}$ columns of S, then the same argument applies.

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References

[1] S. Dăscălescu, L. van Wyk, Do isomorphic structural matrix rings have isomorphic graphs?, Proc. Amer. Math. Soc. 124 (5) (1996) 1385–1391.
[2] A. Hattori, A foundation of torsion theory for modules over general rings, Nagoya Math. J. 17 (1960) 147–158.
[3] M.S. Li, J.M. Zelmanowitz, Artinian rings with restricted primeness conditions, J. Algebra 124 (1989) 139–148.
[4] W.K. Nicholson, On *p.p.* rings, Period. Math. Hungar. 27 (2) (1993) 85–88.
[5] L.W. Small, Semihereditary rings, Bull. Amer. Math. Soc. 73 (1967) 656–658.
[6] K.C. Smith, L. van Wyk, An internal characterisation of structural matrix rings, Comm. Algebra 22 (14) (1994) 5599–5622.
[7] L. van Wyk, Special radicals in structural matrix rings, Comm. Algebra 16 (2) (1988) 421–435.
[8] L. van Wyk, Matrix rings satisfying column sum conditions versus structural matrix rings, Linear Algebra Appl. 249 (1996) 15–28.
[9] S. Veldsman, On the radicals of structural matrix rings, Monatsh. Math. 122 (1996) 227–238.
[10] W. Xue, On *p.p.* rings, Kobe J. Math. 7 (1990) 77–80.